

ON KERNEL OF THE REGULATOR MAP

SEN YANG

ABSTRACT. By using the infinitesimal methods due to Bloch, Green and Griffiths in [1, 4], we construct an infinitesimal form of the regulator map and verify that its kernel is $\Omega_{\mathbb{C}/\mathbb{Q}}^1$, which suggests that Question 1.1 seems reasonable at the infinitesimal level.

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1. BACKGROUND AND QUESTION

Let X be a smooth projective curve over the complex number field \mathbb{C} . In 1970s, Bloch constructed the regulator map $R: K_2(X) \rightarrow H^1(X, \mathbb{C}^*)$ in several ways. Later, Deligne found a different construction by considering $H^1(X, \mathbb{C}^*)$ as the group of line bundles with connections. We recall his construction very briefly as follows.

For x a point on X , we use i_x to denote the inclusion $x \rightarrow X$. The flasque BGQ resolution of $K_2(O_X)$

$$0 \rightarrow K_2(O_X) \rightarrow K_2(\mathbb{C}(X)) \rightarrow \bigoplus_{x \in X^{(1)}} i_{x,*} K_1(\mathbb{C}(x)) \rightarrow 0,$$

shows that $H^0(K_2(O_X))$ can be computed as $\text{Ker}\{K_2(\mathbb{C}(X)) \rightarrow \bigoplus_{x \in X^{(1)}} K_1(\mathbb{C}(x))\}$.

So we have the exact sequence of groups

$$0 \rightarrow H^0(K_2(O_X)) \rightarrow K_2(\mathbb{C}(X)) \rightarrow \bigoplus_{x \in X^{(1)}} K_1(\mathbb{C}(x)).$$

It's known that there exists the following Gysin exact sequence in topology,

$$0 \rightarrow H^1(X, \mathbb{C}^*) \rightarrow H^1(\mathbb{C}(X), \mathbb{C}^*) \rightarrow \bigoplus_{x \in X^{(1)}} \mathbb{C}^*,$$

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where $H^1(\mathbb{C}(X), \mathbb{C}^*) = \varinjlim H^1(X - S, \mathbb{C}^*)$ and S is finite points on X .

The main ingredient to construct the regulator map $R: H^0(K_2(O_X)) \rightarrow H^1(X, \mathbb{C}^*)$ is the following commutative diagram

$$(1.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(K_2(O_X)) & \longrightarrow & K_2(\mathbb{C}(X)) & \longrightarrow & \bigoplus_{x \in X^{(1)}} K_1(\mathbb{C}(x)) \\ \downarrow & & \downarrow R & & \downarrow R & & \cong \downarrow \\ 0 & \longrightarrow & H^1(X, \mathbb{C}^*) & \longrightarrow & H^1(\mathbb{C}(X), \mathbb{C}^*) & \longrightarrow & \bigoplus_{x \in X^{(1)}} \mathbb{C}^*. \end{array}$$

That is, one constructs a map $R: K_2(\mathbb{C}(X)) \rightarrow H^1(\mathbb{C}(X), \mathbb{C}^*)$ and use it to deduce the regulator map $R: H^0(K_2(O_X)) \rightarrow H^1(X, \mathbb{C}^*)$. We refer the readers to [1] and Section 6 in [5] for more details.

This regulator map has nice motivic features and is related with a general program of Bloch-Beilinson conjecture. In this short note, we focus on the following question, see Section 2 in [3] for related discussion. To fix notations, for any abelian group M , $M_{\mathbb{Q}}$ denotes the image of M in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ in the following.

Question 1.1 (Conjecture 2.4 in [3]). *Let $R: H^0(K_2(O_X)) \rightarrow H^1(X, \mathbb{C}^*)$ be the regulator map, then $\text{Ker}(R)_{\mathbb{Q}} = K_2(\mathbb{C})_{\mathbb{Q}}$.*

This question is very difficult to approach, though it has very simple form. For $X = \mathbb{P}^1$, this conjecture has been verified by Kerr [6].

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2. MAIN RESULTS

In this section, we shall define an infinitesimal form of the regulator map $R: H^0(K_2(O_X)) \rightarrow H^1(X, \mathbb{C}^*)$ and verify its kernel is $\Omega_{\mathbb{C}/\mathbb{Q}}^1$. Our approach is inspired by the following result due to Green and Griffiths:

Theorem 2.1 (Page 74 and page 125 in [4]). *Let X be a smooth projective curve over \mathbb{C} , the Cousin flasque resolution of $\Omega_{X/\mathbb{Q}}^1$*

$$0 \rightarrow \Omega_{X/\mathbb{Q}}^1 \rightarrow \Omega_{\mathbb{C}(X)/\mathbb{Q}}^1 \xrightarrow{\rho} \bigoplus_{x \in X^{(1)}} i_{x,*} H_x^1(\Omega_{X/\mathbb{Q}}^1) \rightarrow 0,$$

is the tangent sequence to BGQ flasque resolution of the sheaf $K_2(O_X)$

$$0 \rightarrow K_2(O_X) \rightarrow K_2(\mathbb{C}(X)) \rightarrow \bigoplus_{x \in X^{(1)}} i_{x,*} K_1(\mathbb{C}(x)) \rightarrow 0,$$

where the map ρ is known to take principal parts.

It follows that $H^0(\Omega_{X/\mathbb{Q}}^1)$ can be computed as $\text{Ker}\{\Omega_{\mathbb{C}(X)/\mathbb{Q}}^1 \xrightarrow{\rho} \bigoplus_{x \in X^{(1)}} H_x^1(\Omega_{X/\mathbb{Q}}^1)\}$.

So we have the exact sequence of groups

$$0 \rightarrow H^0(\Omega_{X/\mathbb{Q}}^1) \rightarrow \Omega_{\mathbb{C}(X)/\mathbb{Q}}^1 \xrightarrow{\rho} \bigoplus_{x \in X^{(1)}} H_x^1(\Omega_{X/\mathbb{Q}}^1).$$

Definition 2.2 (page 71 and page 125 in [4]). *For X a smooth projective curve over \mathbb{C} and x a point on X , there exists a residue map*

$$\text{Res} : H_x^1(\Omega_{X/\mathbb{Q}}^1) \rightarrow \mathbb{C},$$

which is defined as follows:

Using $\Omega_{O_{X,x}/\mathbb{Q}}^1(nx)$ to denote the absolute 1-forms with poles of order at most n at x , we define Res_x as the following composition:

$$\Omega_{O_{X,x}/\mathbb{Q}}^1(nx) \longrightarrow \Omega_{O_{X,x}/\mathbb{C}}^1(nx) \xrightarrow{\text{Res}} \mathbb{C}.$$

If ξ is the local uniformizer centered at x , an element of $H_x^1(\Omega_{X/\mathbb{Q}}^1)$ is represented by the following diagram

$$(2.1) \quad \begin{cases} O_{X,x} \xrightarrow{\xi^k} O_{X,x} \longrightarrow O_{X,x}/(\xi^k) \longrightarrow 0 \\ O_{X,x} \xrightarrow{\psi} \Omega_{O_{X,x}/\mathbb{Q}}^1 \end{cases}$$

For such an element, we define $\text{Res}_x(\frac{\psi}{\xi^k}) \in \mathbb{C}$.

It is known that the tangent space to \mathbb{C}^* , which is defined to be the kernel of the natural projection:

$$\mathbb{C}[\varepsilon]^* \xrightarrow{\varepsilon=0} \mathbb{C}^*,$$

can be identified with \mathbb{C} and the tangent map $\tan: \mathbb{C}[\varepsilon]^* \rightarrow \mathbb{C}$ is given by $z_0 + z_1\varepsilon \rightarrow \frac{z_1}{z_0}$. This tangent map further induces a map between cohomology groups $\tan: H^1(X, \mathbb{C}[\varepsilon]^*) \rightarrow H^1(X, \mathbb{C})$. With this interpretation, one can consider $H^1(X, \mathbb{C})$ as the tangent space to $H^1(X, \mathbb{C}^*)$ (this is used in [1]).

There exists the following Gysin exact sequence in topology:

$$0 \rightarrow H^1(X, \mathbb{C}) \rightarrow H^1(\mathbb{C}(X), \mathbb{C}) \rightarrow \bigoplus_{x \in X^{(1)}} \mathbb{C},$$

e.g., see page 54-55 in [2]. The boundary map $H^1(\mathbb{C}(X), \mathbb{C}) \rightarrow \bigoplus_{x \in X^{(1)}} \mathbb{C}$ can be described via Hodge theory as follows. Let $D = \{p_1, \dots, p_n\}$ be finite points on X and let U be the open complement, $U = X - D$. Let $i_D : D \rightarrow X$ denote the inclusion, the residue map $\text{Res}: \Omega_X^\bullet(\log D) \rightarrow i_{D,*} \Omega_D^{\bullet-1}$ induces $\text{Res}: \mathbb{H}^1(\Omega_X^\bullet(\log D)) \rightarrow \mathbb{H}^0(\Omega_D^\bullet)$. This gives the map $\text{Res}: H^1(U, \mathbb{C}) \rightarrow \bigoplus_{i=1, \dots, n} \mathbb{C}$, by using the identifications $\mathbb{H}^1(\Omega_X^\bullet(\log D)) \cong H^1(U, \mathbb{C})$ and $\mathbb{H}^0(\Omega_D^\bullet) = H^0(D, \mathbb{C}) \cong \bigoplus_{i=1, \dots, n} \mathbb{C}$.

The following theorem is an infinitesimal form of diagram (1.1):

Theorem 2.3. *There exists the following commutative diagram*

$$(2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(\Omega_{X/\mathbb{Q}}^1) & \longrightarrow & \Omega_{\mathbb{C}(X)/\mathbb{Q}}^1 & \xrightarrow{\rho} & \bigoplus_{x \in X^{(1)}} H_x^1(\Omega_{X/\mathbb{Q}}^1) \\ & & \downarrow & & \downarrow R' & & \downarrow \text{Res} \\ 0 & \longrightarrow & H^1(X, \mathbb{C}) & \longrightarrow & H^1(\mathbb{C}(X), \mathbb{C}) & \xrightarrow{\text{Res}} & \bigoplus_{x \in X^{(1)}} \mathbb{C}, \end{array}$$

where the map R' 's are the natural maps sending $d_{/\mathbb{Q}}f$ to $d_{/\mathbb{C}}f$.

Proof. The map $R': \Omega_{\mathbb{C}(X)/\mathbb{Q}}^1 \rightarrow H^1(\mathbb{C}(X), \mathbb{C})$ can be described as follows. Let U be open affine in X , $H^1(U, \mathbb{C})$ can be computed as $\Gamma(U, \Omega_{U/\mathbb{C}})/d_{/\mathbb{C}}\Gamma(U, \mathcal{O}_U)$. Given any element $\alpha \in \Omega_{U/\mathbb{Q}}^1$, its image $[\alpha]$ in $\Omega_{U/\mathbb{C}}^1$ defines an element in $H^1(U, \mathbb{C})$.

To check the commutativity of the right square, working locally in a Zariski open affine neighborhood U , we can write an element $\beta \in \Omega_{\mathbb{C}(X)/\mathbb{Q}}^1$ as

$$\beta = \frac{h d_{/\mathbb{Q}}g}{f_1^{l_1} \dots f_k^{l_k}},$$

where $f_1, \dots, f_k, h \in \Gamma(U, \mathcal{O}_U)$ are relatively prime and f_i 's are irreducible.

The following diagram is commutative:

$$\begin{array}{ccc} \frac{h d_{/\mathbb{Q}}g}{f_1^{l_1} \dots f_k^{l_k}} & \xrightarrow{\rho} & \sum_i \frac{h d_{/\mathbb{Q}}g}{f_1^{l_1} \dots \hat{f}_i^{l_i} \dots f_k^{l_k}} \\ \downarrow R' & & \downarrow \text{Res} \\ \frac{h d_{/\mathbb{C}}g}{f_1^{l_1} \dots f_k^{l_k}} & \xrightarrow{\text{Res}} & \sum_i \text{Res}_{x_i} \left(\frac{h d_{/\mathbb{C}}g}{f_1^{l_1} \dots \hat{f}_i^{l_i} \dots f_k^{l_k}} \right), \end{array}$$

where $x_i = \{f_i = 0\}$ and $\hat{f}_i^{l_i}$ means to omit the i^{th} term.

The map $R': \Omega_{\mathbb{C}(X)/\mathbb{Q}}^1 \rightarrow H^1(\mathbb{C}(X), \mathbb{C})$ induces $R': H^0(\Omega_{X/\mathbb{Q}}^1) \rightarrow H^1(X, \mathbb{C})$. \square

Let $\{f_0, g_0\} \in H^0(K_2(O_X))$ and let (N, ∇) denote the bundle with connection ∇ , as recalled on page 4 in [1]. There exists the following commutative diagram:

$$\begin{array}{ccccc}
\{f_0, g_0\} & \xleftarrow{\varepsilon=0} & \{f_0 + \varepsilon f_1, g_0 + \varepsilon g_1\} & \xrightarrow{\tan} & \frac{f_1}{f_0} \frac{d/\mathbb{Q} g_0}{g_0} - \frac{g_1}{g_0} \frac{d/\mathbb{Q} f_0}{f_0} \\
\downarrow \text{R} & & \downarrow & & \downarrow \text{R}' \\
\{f_0, g_0\}^*(N, \nabla) & \xleftarrow[\varepsilon=0]{} & \{f_0 + \varepsilon f_1, g_0 + \varepsilon g_1\}^*(N, \nabla) & \xrightarrow{\tan} & \frac{f_1}{f_0} \frac{d/\mathbb{C} g_0}{g_0} - \frac{g_1}{g_0} \frac{d/\mathbb{C} f_0}{f_0}.
\end{array}$$

The commutativity of left square is trivial. To check the right one, since $\{f_0 + \varepsilon f_1, g_0 + \varepsilon g_1\} = \{f_0, g_0\} \{f_0, 1 + \varepsilon \frac{g_1}{g_0}\} \{1 + \varepsilon \frac{f_1}{f_0}, g_0\} \{1 + \varepsilon \frac{f_1}{f_0}, 1 + \varepsilon \frac{g_1}{g_0}\}$, we reduce to considering $\{1 + \varepsilon f_1, g_0\}$ which is obvious:

$$\begin{array}{ccc}
\{1 + \varepsilon f_1, g_0\} & \xrightarrow{\tan} & f_1 \frac{d/\mathbb{Q} g_0}{g_0} \\
\downarrow & & \downarrow \text{R}' \\
\{1 + \varepsilon f_1, g_0\}^*(N, \nabla) & \xrightarrow{\tan} & f_1 \frac{d/\mathbb{C} g_0}{g_0},
\end{array}$$

where the up tan map is well-known and the down tan map is the formula (2.12) on page 14 in [1].

In this sense, we consider the map $\text{R}' : H^0(\Omega_{X/\mathbb{Q}}^1) \rightarrow H^1(X, \mathbb{C})$ as the infinitesimal form of the regulator map $\text{R} : H^0(K_2(O_X)) \rightarrow H^1(X, \mathbb{C}^*)$ and compute the kernel of R' .

Since $H^1(X, \mathbb{C})$ has Hodge decomposition $H^1(X, \mathbb{C}) \cong H^0(\Omega_{X/\mathbb{C}}^1) \oplus H^1(O_X)$ and the map $\text{R}' : H^0(\Omega_{X/\mathbb{Q}}^1) \rightarrow H^1(X, \mathbb{C})$ naturally maps $d/\mathbb{Q} f$ to $d/\mathbb{C} f$, so R' is the composition $H^0(\Omega_{X/\mathbb{Q}}^1) \rightarrow H^0(\Omega_{X/\mathbb{C}}^1) \hookrightarrow H^1(X, \mathbb{C})$. Hence $\text{Ker}(\text{R}') = \text{Ker}\{H^0(\Omega_{X/\mathbb{Q}}^1) \rightarrow H^0(\Omega_{X/\mathbb{C}}^1)\}$.

Theorem 2.4. $\text{Ker}(\text{R}') = \Omega_{\mathbb{C}/\mathbb{Q}}^1$.

Proof. There is a natural short exact sequence of sheaves

$$0 \rightarrow \Omega_{\mathbb{C}/\mathbb{Q}}^1 \otimes_{\mathbb{C}} O_X \rightarrow \Omega_{X/\mathbb{Q}}^1 \rightarrow \Omega_{X/\mathbb{C}}^1 \rightarrow 0.$$

The associated long exact sequence is of the form

$$0 \rightarrow H^0(\Omega_{\mathbb{C}/\mathbb{Q}}^1 \otimes_{\mathbb{C}} O_X) \rightarrow H^0(\Omega_{X/\mathbb{Q}}^1) \rightarrow H^0(\Omega_{X/\mathbb{C}}^1) \rightarrow H^1(\Omega_{\mathbb{C}/\mathbb{Q}}^1 \otimes_{\mathbb{C}} O_X) \rightarrow \dots$$

So the kernel of $H^0(\Omega_{X/\mathbb{Q}}^1) \rightarrow H^0(\Omega_{X/\mathbb{C}}^1)$ can be identified with $H^0(\Omega_{\mathbb{C}/\mathbb{Q}}^1 \otimes_{\mathbb{C}} O_X) \cong H^0(O_X) \otimes \Omega_{\mathbb{C}/\mathbb{Q}}^1 \cong \mathbb{C} \otimes \Omega_{\mathbb{C}/\mathbb{Q}}^1 \cong \Omega_{\mathbb{C}/\mathbb{Q}}^1$.

□

Since the tangent space to $K_2(\mathbb{C})$ is $\Omega_{\mathbb{C}/\mathbb{Q}}^1$, this theorem suggests that Question 1.1 seems reasonable at the infinitesimal level.

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YAU MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, CHINA

E-mail address: syang@math.tsinghua.edu.cn; senyangmath@gmail.com